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Engineering

Applications of Matching

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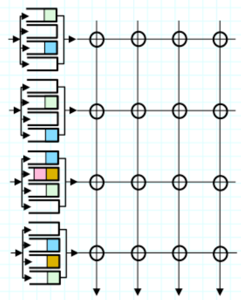
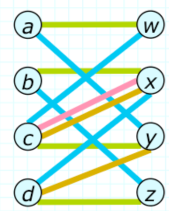
Packet Switch Scheduling

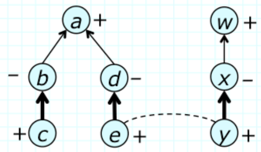
- Internet routers often use "crossbar switches" to transfer packets from inputs to outputs
  - » an input can send one packet at a time, and an output can receive one
  - » packets transferred in one time step define matching in bipartite graph
  - » packets transferred over several time steps define an *edge coloring*
- To find coloring using fewest colors
  - » repeatedly find matching that includes an edge at vertices of maximum degree



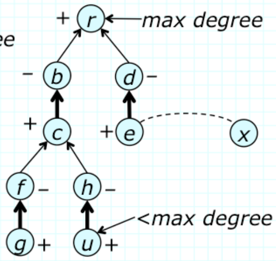


Matching Max Degree Vertices

standard augmenting path method

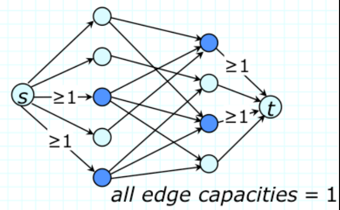


single tree for max degree matching

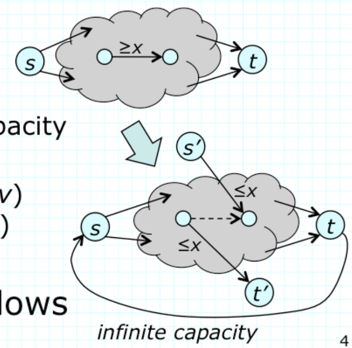


- Find path to extend matching by constructing a single tree rooted at a max degree vertex
  - » if selected edge connects to unmatched vertex, we have an augmenting path and root becomes matched
  - » if selected edge connects to matched vertex, extend tree; if new leaf has <max-degree, swap edges on path to root
    - this does not increase size of matching, but does match root

Alternate Approach



- Construct flow graph for matching as before
  - » augment source/sink edges for max-degree vertices with *minimum flow requirement* of 1
- To find flow that satisfies min flow requirement
  - » find max flow in modified graph
    - add sink/source edge of infinite capacity
    - add new source/sink vertices *s*′, *t*′
    - replace each lower-bound edge (*u*,*v*) with ordinary edges (*s*′,*v*) and (*u*,*t*′)
- Map back to original graph and augment, while retaining min flows



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Observations

- Method using min flows can be used to construct matchings that require specific vertices
  - not just max-degree vertices
- Algorithm applies more generally
  - can be used with arbitrary graphs having arbitrary min flow requirements
  - useful in various application settings
- Not all sets of min flow requirements are feasible
  - given infeasible set of requirements, first phase of algorithm terminates without saturating s' edges
- Other edge coloring methods
  - divide-and-conquer algorithm based on Euler partitions achieves running time of $O(m \log \Delta)$

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Traveling Salesman Problem

- Given a complete graph with edge costs, $c(u,v)$
  - » find min length "tour" that visits every vertex once
- Variants
  - » TSP with triangle inequality – $c(u,w) \leq c(u,v)+c(v,w)$
  - » Euclidean TSP: vertices are points in a plane, there's an edge between every pair with length equal to distance between the points
  - » asymmetric TSP – directed graph with $c(u,v) \neq c(v,u)$
- TSP is NP-complete, but can be approximated
  - » worst-case approx bound of 3/2 with triangle inequality
  - » no bound for asymmetric case, but can get near-optimal solutions with high probability for random instances

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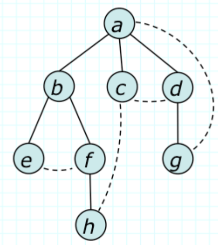
Approximating TSP Using MST

- If we discard an edge from a TSP solution, we get a spanning tree, so
  - » $MST(G) \leq TSP(G)$
- Consider a depth-first traversal of an MST T, from some arbitrary root
  - » list each edge as we go "down" and again as we go back "up"
    - cost of list is $2MST(G)$
  - » select sub-list by replacing repeat edges with "shortcuts"
    - this yields valid TSP tour and if edge lengths satisfy triangle inequality its total length is at most $2MST(G) \leq 2TSP(G)$



ab, be, eb, bf, fh, hf, fb, ba, ac, ca, ad, dg, gd, da

⇓

ab, be, ef, fh, hc, cd, dg, ga

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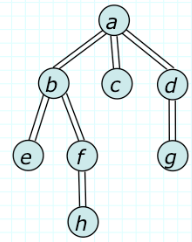
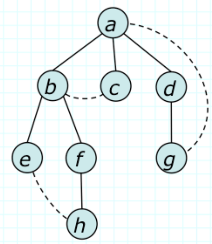
Improving Approximation

- Can view previous procedure as constructing Eulerian graph
  - » where all vertices have even degree
  - » *any* Eulerian graph tour can be converted to a TSP tour using shortcuts
- Finding a better Eulerian graph by connecting odd-degree vertices
  - » by finding a perfect matching in graph induced by odd-degree vertices
    - any graph has an even number of these
  - » min weight perfect matching≤*TSP*(*G*)/2
    - since alternate edges of "shortcut TSP tour" yield two matchings
    - so tour from MST+matching ≤1.5*TSP*(*G*)





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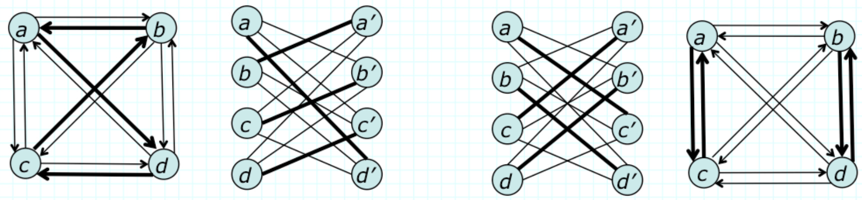
Approximating Asymmetric TSP



- TSP tour is a single cycle spanning all vertices
  - » can view as perfect matching on bipartite graph
- Any perfect matching defines collection of cycles in original graph
  - » so min weight perfect matching provides lower bound on cost of TSP tour
  - » for random edge weights, bound is very tight with high probability

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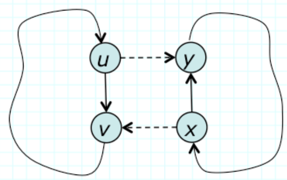
Patching Algorithm for TSP

- Construct weighted bipartite graph and find min cost perfect matching
  - » using min-cost flow method with costs=weights
  - » let *C* be set of cycles defined by matching
- While $|C|>1$
  - » select two cycles and "patch them" using edge pair that produces smallest increase in cost
    - $(c(u,y)+c(x,v)) - (c(u,v)+c(x,y))$



- For random edge weights
  - » initial *C* has small number of cycles
    - with high probability
  - » so small number of patching operations
  - » and small increase in cost, yielding near-optimal TSP tour

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Applications of TSP

- Vehicle routing
  - » selecting route for school bus or mail delivery truck
  - » sub-problem of more general “fleet scheduling”
- Job sequencing
  - » given set of jobs to be carried out on a complex machine tool, where each job requires some setup
    - setup time for one job depends on previous job
  - » use TSP tour to select ordering of jobs to minimize setup
- Data clustering
  - » let $A=[a_{ij}]$ were a_{ij} represents the strength of a relationship between two properties
  - » permute rows and columns to form “high value blocks”
  - » use TSP to find best permutations