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Engineering

Linear Programming and Network Optimization

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Linear Programming

- A linear program seeks values for a set of non-negative real *variables* x_i that optimize a linear function of the x_i subject to linear constraints
  - » maximize $\Sigma_i c_i x_i$ subject to $\Sigma_j a_{ij} x_j \leq b_i$ for all i
  - » or in matrix form, maximize $C^T X$ subject to $AX \leq B$
- Linear programs can be solved efficiently
  - » classical simplex method has exponential worst-case but is fast in practice
  - » interior point method runs in polynomial time
- If some or all of the x_i are constrained to be integers, we get an *Integer Linear Program*
  - » in general, these are *NP*-hard

Max Flow as LP

- Defined over flow variables f_e for each edge e
  - » maximize $\Sigma_{e=(s,u)} f_e$ subject to $0 \le f_e \le cap_e$ for all e and $\Sigma_{e=(w,u)} f_e = \Sigma_{e=(u,v)} f_e$ for all $u \ne s$ or t
  - » to put this into the standard matrix form
    - let F be column vector with an entry per edge
    - let S be a column vector with a 1 entry for every edge leaving the source vertex and a 0 entry for all other edges
    - let I be the identity matrix with m rows and columns
    - let $G=[g_{ue}]$ be an *edge incidence matrix* where for $u \ne s$ or t, $g_{ue}=1$ if u is the tail of e and $g_{ue}=-1$ if u is the head of e
    - define coefficient matrix A by "stacking" I above G above $-G$
    - let B be a column matrix with $m+2(n-2)$ *entries* where first m are the edge capacities and remainder are all 0
    - so LP becomes: maximize $S^T F$ subject to $F \ge 0$ and $AF \le B$

Min Cost Flow as LP

- Also defined over flow variables f_e for each edge e
  - minimize $\Sigma_e cost_e f_e$ subject to $\Sigma_{e=(s,u)} f_e = f^*$ and $0 \leq f_e \leq cap_e$ for all e and $\Sigma_{e=(w,u)} f_e = \Sigma_{e=(u,v)} f_e$ for all $u \neq s$ or t
  - this can also be put into standard matrix form by
    - switching to a maximization problem (maximize $-\Sigma_e cost_e f_e$)
    - expanding coefficient matrix and constraint vector from max flow by adding two rows to express constraint on total flow
- Integrality property for max flow and min cost flow
  - if capacities are integers then optimal flows are also
  - consequence of a general property of coefficient matrix
    - a coefficient matrix is *totally unimodular* if every square sub-matrix has a determinant equal to 0, 1 or −1
  - any LP with integer coefficients and bounds, and a totally unimodular coefficient matrix, has an integral optimum

Multicommodity Flow Problem

- Several types of "stuff" (called commodities) to be moved through a network
  - » can define a separate source and sink for each commodity
  - » each edge can have total flow capacity plus (optional) limits on individual commodity flows
  - » non source/sink nodes must preserve flow of each commodity
- Can be formulated as LP
  - » meaning that it can be solved reasonably efficiently even if we generalize by adding costs, more flow constraints
  - » in general, does not satisfy integrality property
    - coefficient matrix is not totally unimodular
  - » no substantially better solution method than LP

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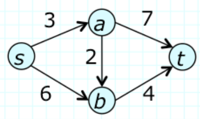
Shortest Path Problem as LP

- For single-source, single-sink version, costs>0
  - » use {0,1} *selection variables* x_e to define path (so ILP)
  - » minimize $\Sigma_e\, cost_e\, x_e$ subject to $\Sigma_{e=(u,t)}\, x_e \geq 1$ and $\Sigma_{e=(w,u)}\, x_e \geq \Sigma_{e=(u,v)}\, x_e$ for all $u \neq s$ or t

minimize $CX = 3x_{sa}+6x_{sb}+2x_{ab}+7x_{at}+4x_{bt}$

subject to $AX \geq B$

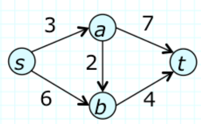


$$\begin{bmatrix} 1 & 0 & -1 & -1 & 0 \\ 0 & 1 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix} \times \begin{bmatrix} x_{sa} \\ x_{sb} \\ x_{ab} \\ x_{at} \\ x_{bt} \end{bmatrix} \geq \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

  - » can also formulate as a min-cost flow problem ($x_e = f_e$)
    - because capacities are all 1, integrality property for min cost flows implies x_e values of an optimal solution are integers
    - so can find optimal solution of shortest path ILP using LP

Alternate LP for Shortest Path

- Imagine a graph as a set of balls connected by strings of different length
  - » pull the source and sink balls as far apart as possible
    - distance separating them is the shortest path distance
- Leads to maximization problem
  - » maximize d_t subject to $d_v \le d_u + cost_{uv}$ for all edges (u,v) and $d_s=0$



maximize d_t
subject to

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & -1 & 1 \end{bmatrix} \times \begin{bmatrix} d_a \\ d_b \\ d_t \end{bmatrix} \le \begin{bmatrix} 3 \\ 6 \\ 2 \\ 7 \\ 4 \end{bmatrix}$$

  - » this is the *dual* of the original LP

Duality

- Standard form LP: maximize $C^T X$ subject to $AX \leq B$
- *Dual*: minimize $B^T Z$ subject to $A^T Z \geq C$ where the vector Z is made up of *dual variables*
- The optimal solution values of the primal and dual forms are equal – $C^T X^* = B^T Z^*$
  - sometimes the dual is easier to solve than primal
- Alternate forms
  - can convert to minimization by negating C
  - can change ≤-bounds to ≥-bounds by negating A and B
  - so for example, if primal expressed as minimize $C^T X$ subject to $AX \leq B$, dual is minimize $B^T Z$ subject to $A^T Z \geq -C$

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Complementary Slackness

- Primal: maximize $C^T X$ subject to $AX \leq B$
  - » $B - AX$ is referred to as *slack* in primal variables
- Dual: minimize $B^T Z$ subject to $A^T Z \geq C$
  - » $A^T Z - C$ is referred to as *slack* in dual variables
- Complementary slackness condition states that X^* and Z^* are optimal solutions if and only if

  $(B - AX^*) = [s_i]$ => $s_i z_j^* = 0$ for all i and

  $(A^T Z^* - C) = [t_j]$ => $t_j x_j^* = 0$ for all j
  - » so each non-zero slack value in primal (dual) corresponds to a zero dual (primal) variable
  - » *primal-dual algorithms* adjust values of primal and dual variables with objective of making these conditions true

9

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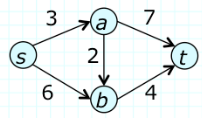
Shortest Path & Complementary Slackness

minimize $CX = 3x_{sa}+6x_{sb}+2x_{ab}+7x_{at}+4x_{bt}$
subject to

$$\begin{bmatrix} 1 & 0 & -1 & -1 & 0 \\ 0 & 1 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix} \times \begin{bmatrix} x_{sa} \\ x_{sb} \\ x_{ab} \\ x_{at} \\ x_{bt} \end{bmatrix} \geq \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$



maximize d_t subject to

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & -1 & 1 \end{bmatrix} \times \begin{bmatrix} d_a \\ d_b \\ d_t \end{bmatrix} \leq \begin{bmatrix} 3 \\ 6 \\ 2 \\ 7 \\ 4 \end{bmatrix}$$

- For primal, optimal solution $X^*=[1\ 0\ 1\ 0\ 1]$
- For dual, optimal solution $D^*=[3\ 5\ 9]$
- Complementary slackness conditions
  - » $(A^TD^*-C)^TX^*=[0]$ and $(B-AX^*)^TD^*=[0]$

$$\left\{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & -1 & 1 \end{bmatrix} \times \begin{bmatrix} 3 \\ 5 \\ 9 \end{bmatrix} - \begin{bmatrix} 3 \\ 6 \\ 2 \\ 7 \\ 4 \end{bmatrix}\right\}^T \times \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\left\{\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} - \begin{bmatrix} 1 & 0 & -1 & -1 & 0 \\ 0 & 1 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix} \times \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}\right\}^T \times \begin{bmatrix} 3 \\ 5 \\ 9 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

10

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Max Matching as ILP

- ILP for maximum size matching problem using 0-1 selection variables $X=[x_e]$
  - » maximize $\Sigma_e x_e$ subject to $\Sigma_{e=\{u,v\}} x_e \le 1$ for all u
  - » to get matrix form, let $G=[g_{ue}]$ be incidence matrix of graph where $g_{ue}=1$ if u is an endpoint of e, else 0
  - » maximize $[1]^T X$ subject to $X \ge 0$ and $GX \le [1]$
- For weighted matching, let W be column vector of edge weights, then
  - » maximize $W^T X$ subject to $X \ge 0$ and $GX \le [1]$
    - can get LP with same optimal solutions by adding constraints
  - » dual: minimize $[1]^T Z$ subject to $Z \ge 0$ and $G^T Z \ge W$
    - the variables z_u can be thought of as vertex labels and the constraints take form $z_u + z_v \ge w_e$ for all $e=\{u,v\}$