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Engineering

Maximum Weight Matchings in General Graphs – Part 1

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Maximum Weight Augmentation

- Given graph $G=(V,E)$ and matching M, define weight of path p to be total weight of its free edges minus total weight of its matched edges
- *Theorem 9.2.* Let M be a matching of maximum weight among matchings of size $|M|$, let p be an augmenting path for M of maximum weight, and let M' be the matching formed by augmenting M using p. Then M' is of maximum weight among matchings of size $|M|+1$.

  Proof. Let M'' be a matching of maximum weight among matchings of size $|M|+1$. Let N be the set of edges in M or M'' but not both.

Define the weight of a path or cycle in N with respect to M.

Any cycle of even length path in N must have weight ≤ 0, since otherwise we could increase the weight of M without changing its size, by exchanging the edges on the cycle or path.

Since N contains exactly one more edge in M'' than in M, we can pair all but one of the odd-length paths so that each pair has an equal number of edges in M and in M''. Each such pair of paths must have total weight ≤ 0 by the same reasoning as before.

Augmenting M using the remaining path gives a matching of size $|M|+1$ with same weight as M''. This must be a maximum weight augmenting path for M since if there were an augmenting path with larger weight, we could construct a matching of size $|M|+1$ with larger weight than M''. ■

- Theorem 9.2 provides a basis for a weighted matching algorithm
  - » finding max weight augmenting paths directly is difficult, especially for general graphs
  - » can be done using LP duality
    - dual variables can be viewed as vertex/blossom labels
    - label values of edge endpoints are related to edge weights

Matching and Linear Programming

- Matchings defined by selection variables $X=\{x_e\}$
  - » $x_e=1$ if e is an edge in the matching
- Objective is to maximize $weight(X)=\Sigma_e\, x_e w(e)$
- Constraints:
  - » for each vertex u with incident edges $E(u)$, $\Sigma_{e\in E(u)}\, x_e \leq 1$
  - » for each edge e, $x_e=0$ or $x_e=1$
- The constraints on the x_es make this an integer linear programming problem
  - » Edmonds showed that for bipartite graphs, we can replace these constraints with $x_e \leq 1$
    - this ordinary LP has same optimal solutions as original ILP
    - we'll use duality to obtain a more efficient algorithm

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Dual Version of Matching LP

- First, re-state primal version in matrix form
  - » define the $n \times m$ *edge incidence matrix* $G=[g_{u,e}]$ where $g_{u,e}=1$ if u is an endpoint of e, else $g_{u,e}=0$
  - » let $W=[w_e]$ be column vector of edge weights and let $X=[x_e]$ be column vector of selection variables
  - » primal problem becomes
    - maximize $weight(X)=W^TX$ subject to $GX \leq [1]$
- Dual version uses variables $Z=[z_u]$
  - » minimize $cost(Z)=[1]^TZ$ subject to $G^TZ \geq W$
  - » equivalently, minimize $\Sigma_u z_u$ subject to $z_u \geq 0$ and for all edges e, $z_e \geq w_e$ where $z_e=z_u+z_v$ for $e=\{u,v\}$
  - » complementary slackness implies that if X^* and Z^* are optimal, $z_e=w_e$ for matching edges e and $z_u=0$ if u is free

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Max Wt. Matchings & Vertex Labeling

- *Theorem.* Let $G=(V,E)$ be a bipartite graph with edge weights $w(e)$, let M be a matching in G and let each vertex u have a non-negative label z_u. If

  (1) $z_e \geq w(e)$ for $e \in E$ $(z_e = z_u + z_v)$

  (2) $z_e = w(e)$ for $e \in M$

  (3) $z_u = 0$ if u is free

  then M is a maximum weight matching.

  Proof. Let M and z satisfy the conditions in the theorem and let N be any other matching.

  $$\Sigma_{e\in N}\, w(e) \leq \Sigma_{e\in N}\, z_e \leq \Sigma_u\, z_u = \Sigma_{e\in M}\, z_e = \Sigma_{e\in M}\, w(e) \quad \blacksquare$$

- Edges with $w(e)=z_e$ are called *equality edges*
  - » augmenting path using equality edges has max weight

Bipartite Matching Using Vertex Labels

- Initialization
  - » M ={} and z_u=(max edge weight)/2 for all u
    - this satisfies conditions (1) and (2) in theorem
- At each step, search for augmenting paths using only equality edges (by building trees, as before)
  - » halt if condition (3) becomes true
  - » if search fails to find an augmenting path, modify labeling
    - this makes condition (3) true or creates more equality edges
    - in latter case, continue search for augmenting path using newly created equality edges
  - » after finding a path, augment and reset even/odd status, but retain z values
    - note, augmentation maintains truth of (1), (2)

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Adjusting Labels

- Whenever the search runs out of eligible edges
  - » if all free vertices have zero labels, terminate
  - » let $\delta_1 = \min \{z_u | u \text{ is even}\}$

    $\delta_2 = \min\{z_e - w(e) \mid e=\{u,v\}, u \text{ even}, v \text{ unreached}\}$

    $\delta_3 = \min\{(z_e - w(e))/2 \mid e=\{u,v\}, \text{both even}\}$

    (δ_2, δ_3 are undefined if no suitable edge)

    $\delta = \min\{\delta_1, \delta_2, \delta_3\}$ – ignore δ_2, δ_3, when undefined
  - » subtract δ from labels for even vertices, add δ to labels for odd vertices
    - note: this maintains truth of (1), (2)
  - » if $\delta=\delta_1$, this makes condition (3) true and algorithm halts
    - since labels start with same value and free vertices experience same sequence of changes
  - » if $\delta=\delta_2$ or δ_3, search can resume using new equality edges

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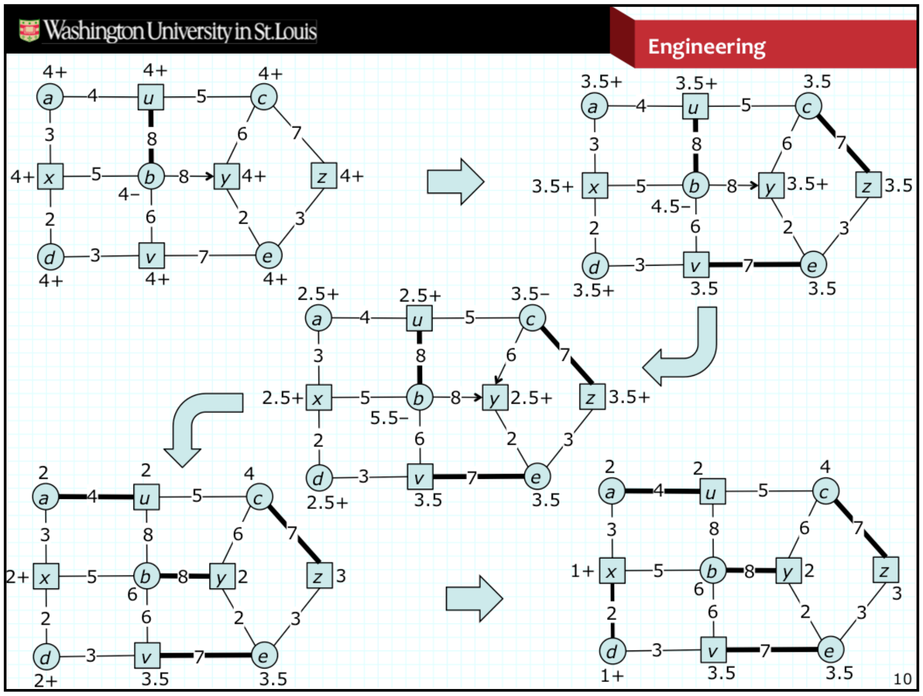
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Implementation Details

- Use heaps to compute δ_i values efficiently
  - » h_{1e} and h_{1o} store the even and odd vertices respectively, with z_u as the key for vertex u
  - » h_2, h_3 store edges, with keys $z_e - w(e)$
    - h_2 has edges with one even endpoint and one unreached, h_3 has edges with both endpoints even
    - when a vertex u becomes even, add its edges to h_2 or h_3
- To enable fast updating of labels use heap with fast *addtokeys*(x) operation
  - » adds x to keys of all items in a heap
  - » d-heap can be extended to do this in constant time
- Eligible equality edges appear at top of h_2 and h_3
  - » can be selected directly from the heaps

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Running Time Analysis

- Number of augmentations is at most $n/2$
  - at end of search, update vertex labels using h_{1e}, h_{1o}
  - also, clear heaps in preparation for next search
- Each step that extends a tree adds edges to heaps and removes edges from heaps
  - during one augmenting search, each edge added to a heap ≤2 times, removed ≤2 times
  - so, $O(mn \log n)$ time for these heap operations
- All but the last label adjustment adds at least one equality edge and does not eliminate any
  - so total # of label adjustments is $O(m)$ and since these require only *findmin* and *addtokeys*, we get $O(m)$ time

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D-Heap with *Addtokeys* Operation

- Operation *addtokeys*(x) adds x to keys of all items in a heap
  - » add internal variable Δ to heap implementation
  - » every *addtokeys*(x) operation increases Δ by x
  - » let $\Delta(t)$ be value of Δ at time t, then
    - from time t_1 to time t_2, *key*(j) increases by $\Delta(t_2)-\Delta(t_1)$
- When inserting item j with key k into heap, use $k-\Delta$ as the stored value, in place of k
  - » preserves relative values of all items in heap
- To obtain the "true key" for item j, add the current value of Δ to the stored key value