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Engineering

Binary Search Trees

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Sorted Sets

- Data structure for a collection of items each having a *distinct* key and belonging to one of several sets
  - » sets are identified by one of their items; initially, each item belongs to a singleton set
- Operations

setkey(i,k): initialize key of item i to k; i must be a singleton

access(k,s): return the item in set s having key k

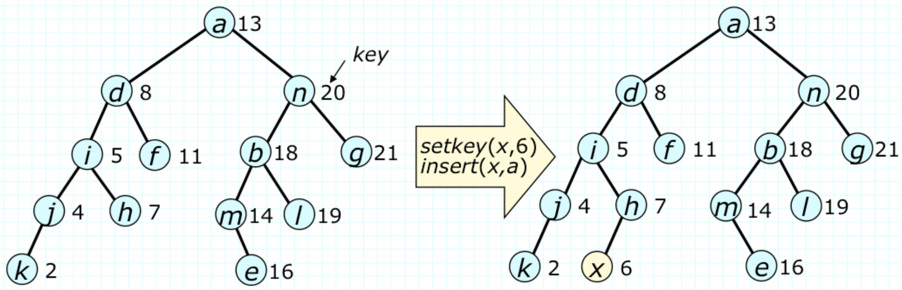
insert(i,s): insert item i into s; i must be a singleton

delete(i,s): remove item i from s; i becomes a singleton

join(s_1,i,s_2) return set formed by combining s_1, i and s_2; all items in s_1 must have keys $<key(i)$ and all items in s_2 must have keys $>key(i)$; operation destroys s_1 and s_2

split(i,s); split set s containing i into three sets: s_1 containing items with keys $<key(i)$, $\{i\}$ and s_2 containing items with keys $>key(i)$; return pair $[s_1,s_2]$

Sorted Sets and Binary Search Trees



- *Symmetric ordering* - items arranged in trees so that for every node *v*
  - » keys of nodes in *v*'s left subtree are smaller than *key*(*v*)
  - » keys of nodes in *v*'s right subtree are larger than *key*(*v*)
- To insert new key value, search for key and insert at place where search "falls out" of tree

item function access(**keytype** *k*, **sorted set** *s*);
    do *s* ≠ **null and** *k*<*key*(*s*) ⇒ *s*:=*left*(*s*)
     | *s* ≠ **null and** *k*>*key*(*s*) ⇒ *s*:=*right*(*s*)
    od;
    return *s*
end;

if any node in tree has key *k*, then subtree rooted at s does

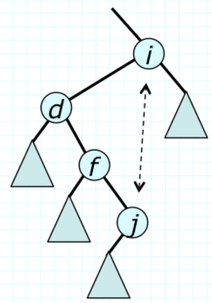
procedure insert(**item** *i*, **sorted set** *s*);
    item *x*; *x*:=*s*;
    if *s* = **null** ⇒ **return** *i* **fi**;
    do *key*(*i*)<*key*(*x*) **and** *left*(*x*) ≠ **null** ⇒ *x*:=*left*(*x*)
     | *key*(*i*)>*key*(*x*) **and** *right*(*x*) ≠ **null** ⇒ *x*:=*right*(*x*)
    od;
    if *key*(*i*)=*key*(*x*) ⇒ **return null**;
     | *key*(*i*)<*key*(*x*) ⇒ *left*(*x*):=*i*;
     | *key*(*i*)>*key*(*x*) ⇒ *right*(*x*):=*i*;
    fi;
    p(*i*) := *x*;
end;

proper insertion location for *i* is in subtree with root *x*

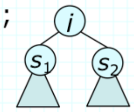
procedure delete(**item** *i*, **sorted set** *s*);
    item *j*;
    if *left*(*i*) ≠ **null and** *right*(*i*) ≠ **null** ⇒
        j := *left*(*i*);
        do *right*(*j*) ≠ **null** ⇒ *j* := *right*(*j*) **od**;
        swapplaces(*i*,*j*);
    fi;
    if *left*(*i*) = **null** ⇒ *left*(*i*) ↔ *right*(*i*) **fi**;
    p(*left*(*i*)) := *p*(*i*);
    if *i* = *left*(*p*(*i*)) ⇒ *left*(*p*(*i*)) := *left*(*i*)
    | *i* = *right*(*p*(*i*)) ⇒ *right*(*p*(*i*)) := *left*(*i*)
    fi;
    left(*i*),*right*(*i*),*p*(*i*) := **null**;
    return *s*
end;

find node *j* with next smaller key

i has <2 children

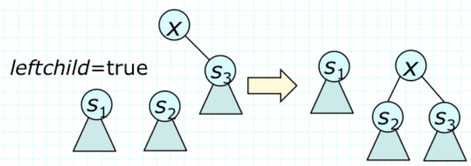


```
sset function join(sset s1, item i, sorted set s2);
    left(i) := s1; right(i) := s2;
    p(s1), p(s2) := i;
    return i;
end;
```




```
[sset, sset] function split(item i, sorted set s)
    sset x,s1,s2;
    x := p(i);  s1,s2 := left(i),right(i);
    leftchild := (i = left(x))
    do x ≠ null ⇒
        if leftchild       ⇒ s2 := join(s2,x,right(x))
        | not leftchild ⇒ s1 := join(left(x),x,s1)
        fi;
        leftchild := (x = left(p(x)); x := p(x)
    od;
    left(i),right(i),p(i) := null;
    p(s1),p(s2) := null;
    return [s1, s2];
end;
```

s_1 (s_2) includes all nodes at or below y that belong in left (right) tree after split



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Analysis of Binary Search Trees

- *Access* takes time proportional to depth of item
- *Insert* takes time proportional to depth of item after insertion
- *Delete* takes time proportional to depth of deleted item if it has a **null** child, and time proportional to depth of its symmetric order predecessor if it has no **null** child
- *Join* take constant time
- *Split* takes time proportional to depth of item on which the split is taking place
- Depth of BST on n nodes can be $n-1$ in worst case, so most operations have worst-case running time $\Omega(n)$
  - » can improve to $O(\log n)$ by *balancing* subtrees

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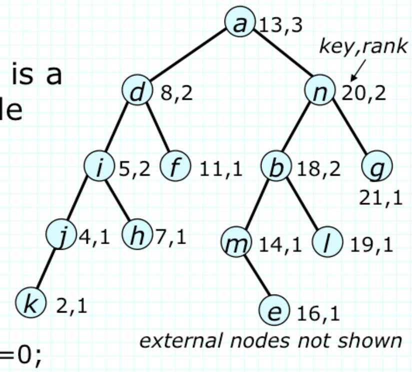
Balanced Binary Trees



- A *balanced binary tree* (BBT) is a full binary tree with each node *x* having integer *rank*(*x*) that satisfies following:
  - » if *x* is a node with a parent, *rank*(*x*)≤*rank*(*p*(*x*))≤*rank*(*x*)+1
  - » if *x* is a node with a grand-parent, *rank*(*x*)<*rank*(*p*(*p*(*x*)))
  - » if *x* is an external node, *rank*(*x*)=0; if *x* also has a parent, *rank*(*p*(*x*))=1
- Also called *red-black trees*
  - » red nodes have same rank as parents, black nodes have different ranks
  - » sufficient to store 1 bit of balance information

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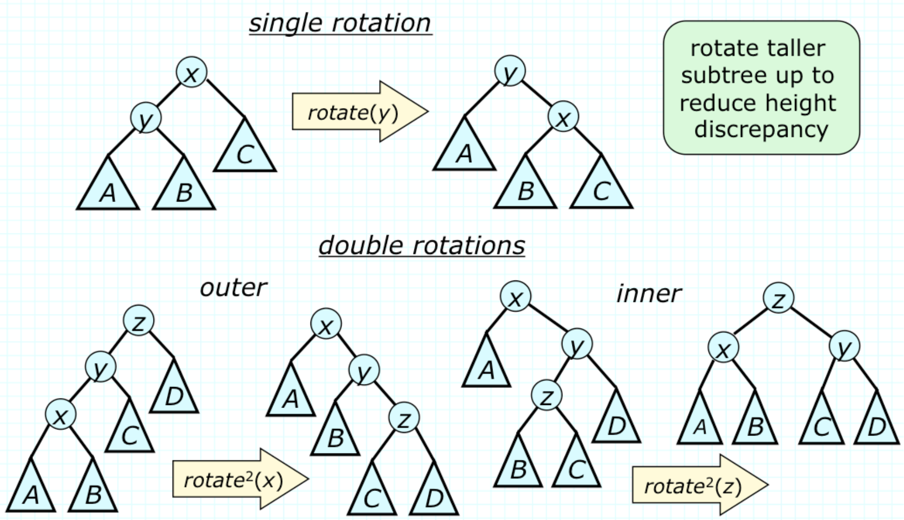
Depth of Balanced Binary Trees

- *Lemma 4.1.* A node of rank k in a balanced binary tree has height at most $2k$ and at least $2^{k+1}-1$ descendants; therefore, a balanced binary tree with n internal nodes has depth at most $2\lg(n+1)$

  Proof. The proof of first part is by induction on k
  - » basis (k=0) is obvious since by definition of ranks, any node of rank 0, must be external, hence its height is 0 and it has 1 descendant
  - » assume lemma is true for rank k, and let x have rank k+1
    - by definition of ranks and induction hypothesis, the grandchildren of x have height at most $2k$, so x can have height at most $2(k+1)$
    - similarly, its two subtrees must contain at least $2^{k+1}-1$ nodes, so x has a total of at least $2(2^{k+1}-1)+1=2^{k+2}-1$ descendants

  by the first part of the lemma, the rank of the root is at most $\lg(n+1)$ and the height of the root is at most twice its rank ■
- Lemma implies access time in a BBT is $O(\log n)$

Rotation Operations



rotate taller subtree up to reduce height discrepancy

Convenience Notations/Operations

- Special notations
  - » $p^2(x)$ – $p(p(x))$
  - » *sib*(x) – sibling of x
  - » *uncle*(x) – child of grandparent that is not parent
  - » *nephew*(x) – far child of sibling
  - » *niece*(x) – near child of sibling
  - » *outer*(x) – true if x is leftmost or rightmost grandchild
    - *inner*(x) – not *outer*(x)
- Operations
  - » *rotate*(x) – rotates x up to parent's position
  - » *rotate*^{2}(x) – rotates x up to grandparent's position
    - equivalent to *two rotations*

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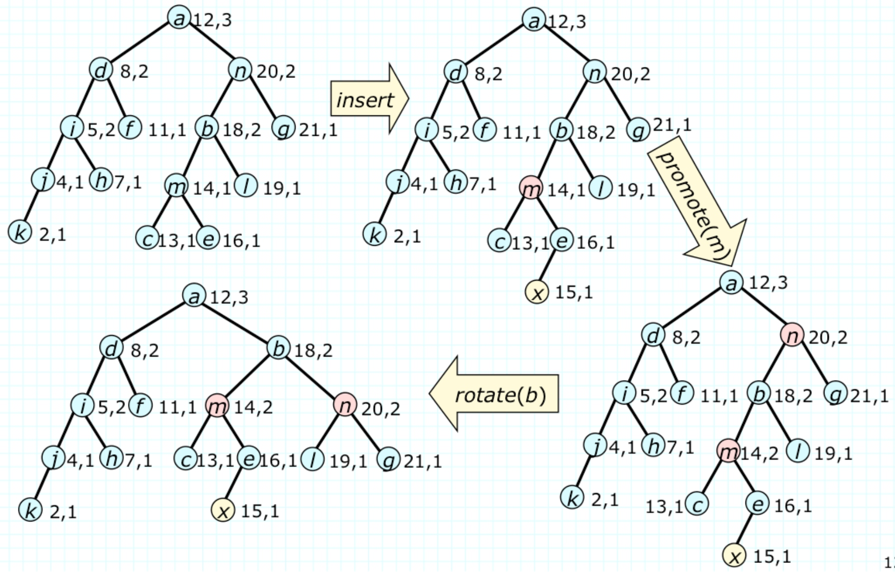
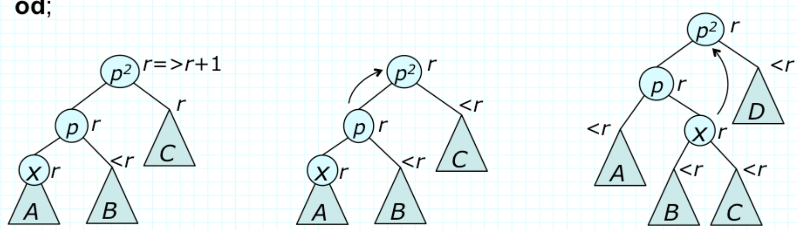
Insertion in BBTs



Rebalancing After Insert/Delete

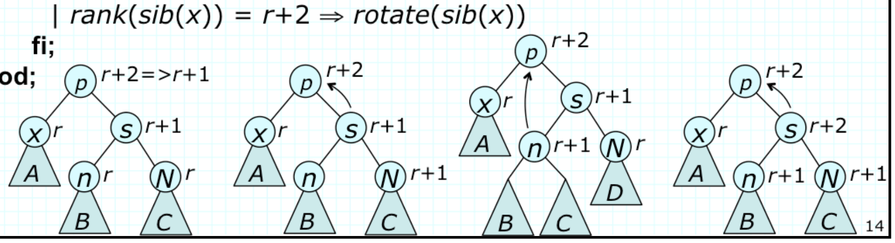
In case of insertion, let *x* be inserted node

```
do p²(x) ≠ null and rank(x) = rank(p²(x)) ⇒
    if rank(p(x)) = rank(uncle(x)) ⇒
        x := p²(x); rank(x) :=  rank(x) + 1;
     | rank(p(x)) ≠ rank(uncle(x)) ⇒
        if outer(x) ⇒ x := rotate(p(x)); done
         | inner(x) ⇒ x := rotate²(x); done
        fi;
   fi;
od;
```



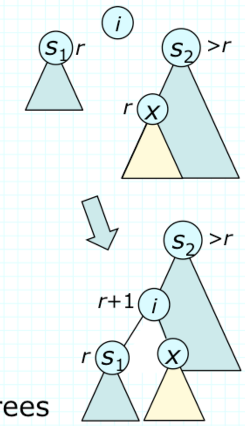
In case of deletion, let *x* be root of the subtree that moved up (if any)

```
do p(x) ≠ null and rank(p(x))=rank(x)+2 ⇒
   r := rank(x)
   if rank(sib(x)) = r+1 ⇒
       if rank(nephew(x)) = rank(niece(x)) = r ⇒
           x := p(x); rank(x) := r+1;
        | rank(nephew(x)) = r+1 ⇒
           rotate(sib(x)); rank(p(x)) = r+1; rank(p^2(x))=r+2; done
        | rank(niece(x)) = r+1 > rank(nephew(x)) ⇒
           rotate^2(niece(x)); rank(p(x))=r+1; rank(p^2(x))=r+2; done
       fi;
     | rank(sib(x)) = r+2 ⇒ rotate(sib(x))
   fi;
od;
```



Join and Split

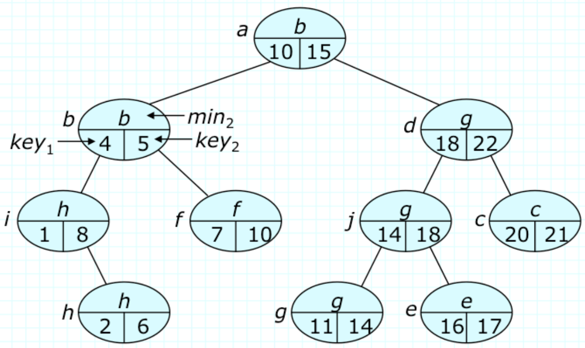
- *join*(s_1,*i*,s_2): when $rank(s_1) < rank(s_2)$
  - » find leftmost node *x* in s_2 with $rank(x)=rank(s_1)$
  - » replace *x* with *i*, make s_1 the left child and *x* the right child of *i*
  - » make $rank(i)=rank(s_1)+1$ then rebalance as in insert
- *split*(*i*,*s*) is done in same way as for unbalanced trees but using new join
  - » differences between ranks of joined subtrees sums to $\leq rank(root)$, so $O(\log n)$ time
- Can use to implement split/join lists
  - » $O(\log n)$ time per operation



Dual Key Search Trees

- Data structure for sets of items having two keys key_1 and key_2; key values need not be unique
  - » *setkey*(i,k_1,k_2): initialize keys of singleton item i to k_1,k_2
  - » *insert*(i,s): insert i into s
  - » *delete*(i,s): remove item i from s; i becomes a singleton
  - » *access*(k_1,s): return some item in set s having key_1=k_1
  - » *findmin*(k_1,s): return item in s with smallest key_2 value among those with $key_1 \leq k_1$
- Combines aspects of search trees and heaps
- Can implement all ops in $O(\log n)$ time
- Extend to find min key_2 item in key_1 range [*lo*,*hi*]
  - » can use to find min key_2 item in a time interval

Implementation



- Nodes arranged in binary search tree order on key_1
- Field min_2 is node in subtree with smallest key_2 value
- After insertion/deletion, update min_2 fields going up tree
- Also, update during rotation operations

Implementing *findmin*

let key_2(**null**) be larger than any valid key, let min_2(**null**)=**null**

```
item function findmin(keytyp k1, sset2 s);
    item u, v; u := s; v := null;
    do u ≠ null ⇒
        if key1(u) > k1 ⇒
            u := left(u);
        | key1(u) ≤ k1 ⇒
            if key2(u)<key2(v) ⇒ v := u; fi;
            if key2(min2(left(u))) < key2(v) ⇒ v := min2(left(u)) fi;
            u := right(u);
        fi;
    od;
    return v;
end;
```

v is best node found so far

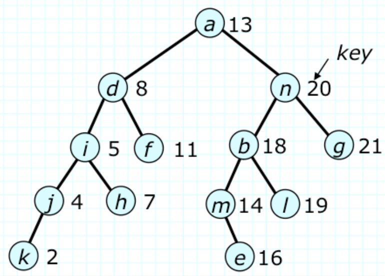
key_1 values of *u* and *right*(*u*) are too large, so "target" must be in *left*(*u*)

all nodes in *left*(*u*) are "eligible" so no need to search *left*(*u*)

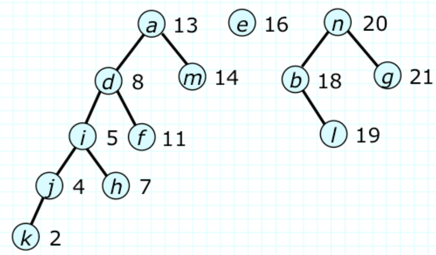
right(*u*) might contain an eligible node

Exercises

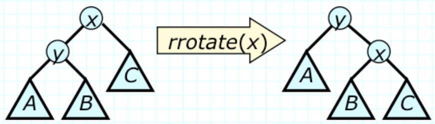
1. Show the result of performing a split operation at node *e* on the (unbalanced) binary search tree shown below.



The split produces the three trees shown below.

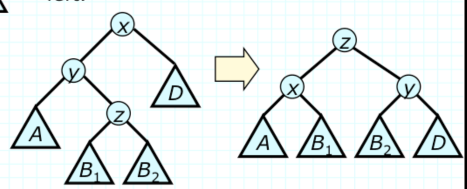


2. The diagram below shows a right rotation operation at node *x*. Show that if the *height* of subtree *A* is larger than the heights of subtrees *B* and *C*, the rotation reduces the height of the overall tree.



The height of a tree is the max distance from its root to one of its leaves, so in the left hand tree, the overall height is 2+height(A), if the height of A is larger than the heights of B and C. The height of the right hand tree, is max(1+height(A),2+height(B), 2+height(C)) but since A has height greater than the other two, this is 1+height(A).

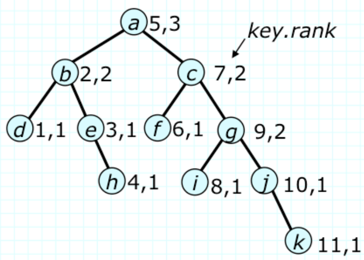
Now suppose that in the previous diagram, the height of *B* is larger than the heights of subtrees *A* and *C*. The diagram at left below shows the same tree but with the subtree *B* shown in more detail (with root *z* and its subtrees B_1 and B_2). The right hand diagram is obtained from the left by a double rotation. Show that the right-hand tree has smaller height than the left.



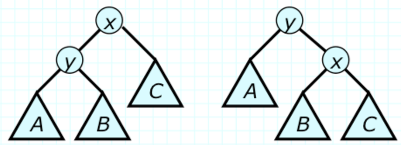
The height of the left-hand tree is 3+max(height(B_1),height(B_2)) and the height on the right is 2+max(height(B_1),height(B_2))

3. Draw a picture of a balanced binary tree on 10 nodes that is as unbalanced as it can possibly be and show a sequence of insert and delete operations that will produce that tree.

   The diagram below has 11 nodes. If the nodes are inserted in alphabetical order, we get the balanced binary tree shown below. If we then remove node h, there is no further change in the tree structure and we get a maximally unbalanced 10 node tree.



4. Consider a rotation operation at *y* in the dual key search tree shown below. How would you update the min_2 values for nodes *x* and *y* shown below?



   With the assignment
   $min_2(x) := min(key_2(x), min_2(B), min_2(C))$
   followed by
   $min_2(y) := min(key_2(y), min_2(A), min_2(x))$
   .

5. Write a program for the dual key search tree that implements an operation *findminRight*(k_1,*s*) that returns the item with the smallest key_2 value in *s* from among those items with key_1 values that are $\geq k_1$.

```
item function findminRight(keytyp k1, sset2 s);
    item u, v; u := s; v := null;
    do u ≠ null ⇒
        if key1(u) < k1 ⇒
            u := right(u);
        | key1(u) ≥ k1 ⇒
            if key2(u) < key2(v) ⇒ v := u; fi;
            if key2(min2(right(u))) < key2(v) ⇒
                v := min2(right(u));
            fi;
            u := left(u);
        fi;
    od;
    return v;
end;
```

How would you extend the data structure so that it would also support a *findmax* operation?

Add a max_2 *field to each node that identifies the node in the subtree of the given node that has the largest* key_2 *value. These fields can be updated in the same way as the* min_2 *fields.*