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Engineering

Fibonacci Heaps

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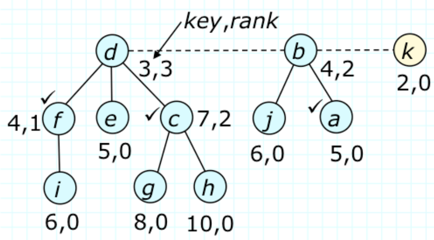
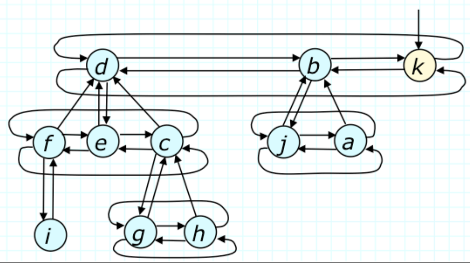
Fibonacci Heaps

- Collection of *meldable* heaps
  - » *meld* operation combines two heaps
  - » each heap is identified by one of its members (its *id*)
  - » initially, all items form singleton heap
  - » good *amortized running time*
- Heap operations
  - »*findmin*(h): return an item of minimum key in (heap with id) h
  - »*insert*(i,x,h): insert item i into heap h with key x
    - i must be a singleton heap
  - »*delete*(i,h): delete item i from h and return resulting heap's id
  - »*deletemin*(h): delete a min key item from h; return it and new id
  - »*meld*(h_1,h_2): return id of heap formed by combining h_1 and h_2; operation destroys h_1 and h_2
  - »*decreasekey*(Δ,i,h): decrease key of i in h by Δ; return new id

Structure of Fibonacci Heaps

- Each F-heap is represented by a *collection* of heap-ordered trees
  - »each node has its item's *key*, an integer *rank* and a *mark* bit
    - *rank*(*i*) equals the number of children of *i*
  - »each node has pointers to its parent, its left and right siblings and one of its children
  - »the tree roots are linked together on a circular list
  - »heap is identified by a root node of minimum key

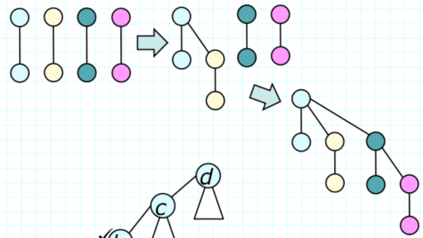




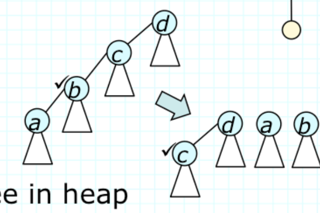
Implementing F-Heap Operations

- For *meld*, combine root lists; implement *insert* as *meld*
  - »new heap identified by item of minimum key; takes $O(1)$ time
- For *delete*(i,h)
  - »perform a *decreasekey* at i, to make i the item with smallest key
  - »perform a *deletemin* to remove i from the heap
  - »restore original key value of i
  - »time is just sum of times for *deletemin* and *decreasekey*
- For *deletemin*
  - »remove min key item from root list
  - »combine its list of children with root list and clear mark bits of children
  - »find new min key node
    - while doing this, combine trees with root nodes of equal rank until no two nodes in root list have same rank

- *Deletemin* combines trees with equal rank roots
  - » insert tree roots into an array, at position determined by their rank
  - » make one root a child of the other whenever there is a "collision"
    - • note that root of new tree increases its rank



- For *decreasekey*(Δ,*i*,*h*)
  - » subtract Δ from *key*(*i*) then *cut* edge joining *i* to its parent *p*
  - » make detached subtree a separate tree in heap and clear its mark bit
  - » if *key*(*i*)<*key*(*h*), *i* becomes the min node of heap
  - » if *p* is not a tree root, and *i* is second child cut from *p*, since *p* became child of some other node, *cut* edge from *p* to its parent
    - • apply this rule recursively to parent of *p*, then its parent,...
    - • use mark bit to identify nodes that have lost a child
  - » increases number of trees, decreases number of marked nodes



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Amortized Analysis

- Objective is to bound total time for sequence of ops
  - »some individual ops may take more time than others
  - »expensive ops must be balanced by (earlier) inexpensive ops
- To facilitate analysis, imagine we're given *credits* for each operation
  - »one credit pays for one unit of computation
  - »credits not used to pay for a current op can be saved for later
  - »the credit allocation for each operation is its effective cost
- Central question: "How many new credits needed for each op to ensure there are always enough on hand?"
- Following *credit invariant* is key to analysis

  at all times, the number of credits on hand is at least the number of trees in all heaps, plus twice number of marked non-root nodes

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- Determine number of new credits needed per op to pay for the op and maintain validity of invariant
  - *findmin*, *insert* and *meld* each take constant time and don't affect invariant, so just one new credit for each op
  - time for *deletemin* bounded by number of steps in second part
    - so, need one new credit per step plus one for every net new tree
    - details to come
  - time for *decreasekey* bounded by number of cuts performed and each cascading cut involves a marked node
- Detailed analysis of *decreasekey*
  - let k=number of cuts made by *decreasekey*
  - running time for *decreasekey* is $O(k)$
  - number of trees increases by k
  - number of marked non-root nodes *decreases* by k–2
  - so, the number of new credits needed is $k+k-2(k-2)=4$
  - so, cost of the *decreasekey* is $O(1)$

Detailed Analysis of Deletemin

- Detailed analysis of *deletemin*
  - » let k=rank of node removed in *deletemin*
    - number of trees increases by k during first part of the op
    - number of marked non-root nodes does not increase
  - » in second part, trees with roots of equal rank are combined
  - » let p=# of times a tree root collides with another, let q=# of times a tree root is inserted with no collision
    - running time for *deletemin* is $O(p+q)$
    - number of trees decreases by p during the second part
  - » so, number of new credits needed to pay for the op and maintain credit invariant is $(p+q)+(k-p)=k+q$
  - » note that both k and q are bounded by the max rank, which we will show is $O(\log n)$
- So, $O(s+t\log n)$ time for s *findmin*, *meld* or *decreasekey* ops plus t *delete* or *deletemin* ops

Bound on Ranks

- *Lemma 1*. Let x be any node and let $y_1,\ldots,y_r$ be children of x, in order of time in which they were linked to x (earliest to latest); then, $rank(y_i) \geq i-2$ for all i

  Proof. Just before y_i was linked to x, x had at least $i-1$ children

  So at that time, $rank(y_i)$ and $rank(x)$ were equal and $\geq i-1$

  Since y_i is still a child of x, its rank has been decremented at most once since it was linked, implying $rank(y_i) \geq i-2$ ■

- *Corollary 1*. A node of rank k has $\geq F_{k+2} \geq \phi^k$ descendants (including itself), where F_k is k-th Fibonacci number, defined by $F_0=0$, $F_1=1$, $F_k=F_{k-1}+F_{k-2}$ and $\phi=(1+5^{1/2})/2$

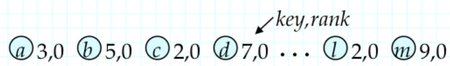
  Proof. Let S_k be min possible number of descendants of a node of rank k; clearly, $S_0=1$, $S_1=2$ and by Lemma 1, $S_k \geq 2+\Sigma_{0\leq i\leq k-2}\, S_i$ for $k\geq 2$; the Fibonacci numbers satisfy $F_{k+2}=1+\Sigma_{0\leq i\leq k}\, F_i$ from which $S_k \geq F_{k+2}$ follows by induction on k ■

  Corollary implies that $rank(x)$ is $O(\log n)$

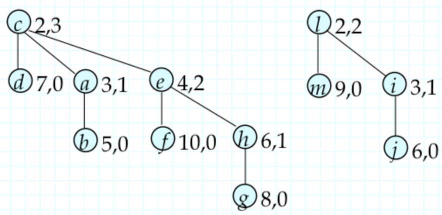
Exercises

1. Assume that items *a* through *m* with keys 3, 5, 2, 7, 4, 10, 8, 6, 3, 6, 1, 2, 9 are inserted in alphabetical order into a Fibonacci heap. Show the heap following the insertions. Then do a *deletemin* and show the resulting heap state.

   Data structure after insertions (single node trees linked in circular list



   Data structure after deletemin (including linking process).



2. Let $P_d(n)$ denote the running time of Prim's algorithm using *d*-heaps, where the value of *d* is chosen dynamically to give the best overall running time. Let $P_F(n)$ denote the running time of Prim's algorithm, using Fibonacci heaps. Which of the following statements is true? Justify your answers.

   P_d is $O(P_F)$ when $m = 3n$.

   This is true, since $P_d = O(m (\log n)/\log(2+m/n)) = O(n \log n)$ *and* $P_F = \Omega(m + n \log n) = \Omega(n \log n)$.

   P_d is $O(P_F)$ when $m = n^2/4$.

   This is true, since $P_d = O(m (\log n)/\log(2+m/n)) = O(n^2)$ *and* $P_F = \Omega(m + n \log n) = \Omega(n^2)$.

   P_d is $O(P_F)$ when $m = n (\log n)^2$.

   This is false, since $P_d = \Omega(m (\log n)/\log(2+m/n)) = \Omega(n (\log n)^3 / \log \log n)$ *and* $P_F = O(m + n \log n) = O(n (\log n)^2)$ *and* $n (\log n)^3 / \log \log n$ *grows more quickly than* $n (\log n)^2$ *does.*

   P_d is $O(P_F)$ when $m = n^{3/2}$.

   This is true, since $P_d = O(m (\log n)/\log(2+m/n)) = O(n^{3/2})$ *and* $P_F = \Omega(m + n \log n) = \Omega(n^{3/2})$.

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3. In the Fibonacci heaps data structure, a cut between a vertex *u* and its parent *v* causes a cascading cut at *v* if *v* has already lost a child since it last became a child of some other vertex. Suppose we change this, so that a cascading cut is done at *v* only if *v* has already lost *two* children. How does this change alter the lemma shown below (this lemma is from the analysis of the running time of Fibonacci heaps)? Explain your answer.

   Lemma. Let x be any node in an F-heap. Let $y_1, \ldots, y_r$ be the children of x, in order of time in which they were linked to x (earliest to latest). Then, $rank(y_i) \geq i-2$ for all i.

   The inequality in the lemma becomes $rank(y_i) \geq i-3$. Since y_i had the same rank as x when it became a child of x and x must have had at least i-1 children at that time, y_i must have had rank of at least i-1 when it became a child of x. Since it still is a child of x, it can have lost at most two children since that time, so its rank must be at least i-3.

   Let S_k be the smallest possible number of descendants that a node of rank k has, in our modified version of Fibonacci heaps. Give a recursive lower bound on S_k. That is, give an inequality of the form $S_k \geq f(S_0, S_1, \ldots, S_{k-1})$ where f is some function of the S_i's for $i<k$.

   Clearly $S_0=1$, $S_1=2$ and $S_2=3$. For k>2, we can use the modified lemma to conclude that $S_k \geq 3+S_0+S_1+\ldots+S_{k-3}$. Note that the difference between the bounds for S_k and for S_{k-1} is S_{k-3}.

   Use this to give a lower bound on the smallest number of descendants that a node with rank 7 can have.

   From the above, we have $S_3 \geq 3+S_0=4$, $S_4 \geq 4+S_1=6$, $S_5 \geq 6+S_2=9$, $S_6 \geq 9+S_3 \geq 13$, $S_7 \geq 13+S_4 \geq 19$.