# The Max Flow Problem

Jonathan Turner

February 6, 2013

Let $G = (V, E)$ be directed graph with a *source* vertex $s$, a *sink* vertex $t$ and a positive real *capacity* $cap(u, v)$ for every edge $(u, v)$ ($cap(u, v) = 0$ if there is no edge $(u, v)$). We define a *flow* on $G$ to be a real-valued function $f$ on vertex pairs that satisfies the following properties:

- *Skew symmetry*: $f(u, v) = -f(v, u)$
- *Capacity constraint*: $f(u, v) \leq cap(u, v)$
- *Flow conservation*: for every vertex $u$ except $s$ and $t$, $\sum_v f(u, v) = 0$

An example of a flow is shown in Figure 1.

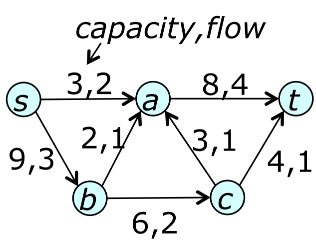


Figure 1: Example of a flow

In the *maximum flow problem*, the objective is to find a flow of maximum value, where the *value* $|f|$ of a flow $f$ is $\sum_v f(s, v)$. So, the value of the flow in Figure 1 is 5 and a maximum flow for this graph has value 11.

We define a *cut* to be a partition of $V$ into two parts $X$, $X'$ with $s \in X$ and $t \in X'$. The *capacity* $cap(X, X')$ of the cut is defined as $\sum_{u \in X, v \in X'} cap(u, v)$. A cut of minimum capacity is called a *minimum cut*. Figure 2 shows a cut with capacity 11. The *flow across a cut* $f(X, X') =$

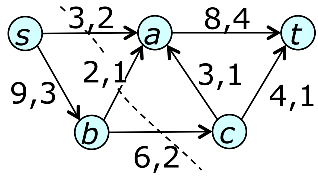


Figure 2: Example of a minimum cut

$\sum_{u \in X, v \in X'} f(u,v)$, so in Figure 2, flow across the cut is 5. We can show that the flow across across any cut is equal to the value of the flow.

$$
\begin{aligned}
f(X,X') &= \sum_{u \in X, v \in X'} f(u,v) \\
&= \sum_{u \in X, v \in V} f(u,v) - \sum_{u \in X, v \in X} f(u,v) \\
&= \sum_{v} f(s,v) + \sum_{u \in X - \{s\}, v \in V} f(u,v) - \sum_{u \in X, v \in X} f(u,v) \\
&= |f|
\end{aligned}
$$

In the third line above, the middle summation is zero, by the flow conservation property and the last summation is zero by the skew symmetry property. This gives us the following lemma.

**Lemma 1** *For any flow $f$ and cut $X, X'$, $|f| = f(X,X')$.*

Next, we define the *residual capacity* $\text{res}(u,v) = \text{cap}(u,v) - f(u,v)$. This represents the amount by which we can add flow from $u$ to $v$ without violating the capacity constraint. So in Figure 2, $\text{res}(s,a) = 1$ and $\text{res}(c,b) = 2$. The *residual graph* $R_f$ with respect to a flow $f$ is a graph with an edge $(u,v)$ for every pair of vertices for which $res(u,v) > 0$. The capacity of an edge in the residual graph is defined to be its residual capacity with respect to $f$. Figure 3 shows the residual graph that corresponds to the flow in Figure 2. Note that if there is a path from $s$ to $t$ in the residual graph for $f$, then we can increase the value of the flow, by adding flow along this path. We call such a path an *augmenting path* with respect to a flow $f$. The *residual capacity of a path p* is defined as $res(p) = \min_{(u,v) \in p} res(u,v)$. Note that the flow on an augmenting path $p$ can be increased by $res(p)$.

We're now ready to show that the value of a maximum flow in a graph is equal to the capacity of a minimum cut. We start by observing that if

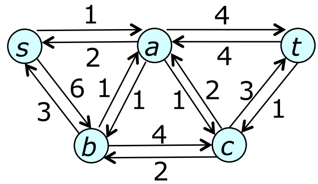


Figure 3: Residual graph

$f$ is a maximum flow, there can be no augmenting path with respect to $f$ (since if there were, we could increase the value of $f$).

Next, note that if there is no augmenting path with respect to $f$, then if we define $X$ to be the set of vertices reachable from $s$ in $R_f$, then $X, X' = V - X$ is a cut for which

$$cap(X, X') = \sum_{u \in X, v \in X'} cap(u, v) = \sum_{u \in X, v \in X'} f(u, v) = |f|$$

That is, the capacity of the cut is equal to the value of the flow. This is illustrated in Figure 4. In this example, the residual graph is shown at right.

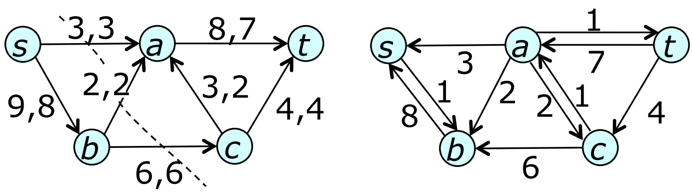


Figure 4: Max flow, min cut

Note that the set of vertices reachable from $s$ in $R$ is $X = \{s, b\}$ and the edges $(s, a)$, $(b, a)$ and $(b, c)$ cross the cut from $X$ to $\{a, c, t\}$, which has a capacity of 11.

Finally, we note that if $f$ is a flow whose value is equal to the capacity of some cut, then $f$ must be a maximum flow, since the flow value cannot exceed the capacity of any cut. These observations are summarized in the *max flow, min cut theorem.*

**Theorem 1** *The following statements are equivalent.*

1. *$f$ is a maximum flow*

2. *there is no augmenting path with respect to $f$*

*3. there is some cut* $X, X'$ *with* $cap(X, X') = |f|$

The theorem leads directly to the *augmenting path method* for finding maximum flows. This method starts with a zero flow, then repeats the following step as long as possible.

> *Augmenting step.* Find an augmenting path $p$ with respect to the current flow and add $res(p)$ units of flow to every edge on this path.

Note that if the edge capacities are integers, then each step increases the flow by an integral amount. Hence if all edge capacities are integers, there is a maximum flow for which all the flow values are integers.

Also note that each step of the augmenting path method increases the value of the flow by at least 1 and so the number of steps is at most $|f^*|$ where $f^*$ is a maximum flow. Figure 5 shows that, the number of steps required can be this bad, if we alternate between the paths $s, a, b, t$ and $s, b, a, t$. Turning

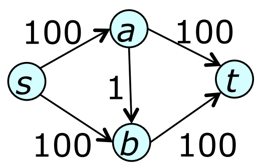


Figure 5: Worst-case for general augmenting path method

the augmenting method into an efficient algorithm depends critically on how we select paths.